Application of Variational Calculus in Radiosity Method

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Abstract:

The original radiosity method searches for the radiosity distribution in a piecewise constant function form. Using this stepwise constant assumption about the radiosity distribution, the integral equation describing the energy transfer is transformed to a linear equation system. Higher order radiosity method means the approximation of the radiosity distribution by more complex functions, as for example, by piecewise linear, harmonic, wavelet, etc. function series with unknown coefficients. Due to higher order approximation, the number of the unknown variables can be significantly smaller than the number of constant steps in the original method. This paper discusses the conversion of the integral equation to an equivalent variational problem which can result in a linear equation system for the unknown coefficients. Three function bases are examined in detail in this framework: piecewise constant, piecewise linear and harmonic approximations.

Keywords:

Radiosity method, integral equations, Ritz's method.

Introduction

The original radiosity method is based on finite element techniques [SF73]. It establishes an integral equation describing the diffuse interreflections of surfaces, and this integral equation is transformed to a linear equation system after decomposing the surfaces into finite, planar surface elements, supposing that these surface elements are homogenous in terms of radiosity, emission and diffuse coefficients [FG85][GCT86]. In other words, the radiosity distribution is searched for in a piecewise constant function form, reducing the original problem to the calculation of the values of the constant steps.

The idea of piecewise constant approximation is theoretically simple and easy to accomplish, but an accurate solution would require a large number of steps making the solution of the linear equation difficult. Furthermore, constant approximation can introduce unexpected artifacts in the picture even if it is smoothed by Gouraud shading in rendering phase.

Recently several papers have been published about the higher order radiosity approximation [Zat93][SGCH94][SK93] which used more complex functions. There are basically two different techniques that can be used to address the problem of higher order solution of integral equations: Galerkin's method and Ritz's method. Both of them approximate the function-to-be-found by a function series using some predefined set of basis functions and tries to find the unknown coefficients in this series by establishing a system of linear equation for them. In Galerkin's method the unknown coefficients are calculated to make the error function (residuum) perpendicular to the subspace spanned by the basis vectors. Ritz's method, on the other hand, creates a variational problem, that is, it establishes a functional which is extreme for a function if and only if the function satisfies the original integral equation. Then it finds this minimum by calculating the partial derivatives of the functional (which is now a function of the unknown coefficients) and making them equal to zero. The definition of an equivalent functional is possible if the operator in the equation is linear, positive and symmetric (exact definitions are coming later).

From functional analysis point of view, Galerkin's method is a generalization of Ritz's method and does not require the operator of the integral equation to be linear, positive and symmetric, unlike Ritz's method. However, Ritz's method is always convergent, but proving the convergence of Galerkin's method is usually a difficult problem. Therefore it is worth selecting Ritz's method if the operator involved makes that possible.

Unfortunately the radiosity equation is not suitable for Ritz's solution in its original form, thus it is not surprising that most of the papers examining higher order radiosity approximation used Galerkin's method [Zat93][SGCH94][DM85].

This paper demonstrates that the application of Ritz's variational approach is also viable, just the radiosity equation must be transformed to make it appropriate for the method. A short introduction is given about Ritz's approach to solve integral equations, and its variational aspects are also highlighted. In order to demonstrate the application of the method, three types of function bases are discussed: piecewise constant functions which lead to the traditional method, piecewise linear functions and harmonic (cosine) functions.

Variational solution of integral equations

Let us suppose that an integral equation is given in the following form:

$$\mathcal{L}f(p) = g(p) \tag{1}$$

where $\mathcal{L}$ is a linear operator and g is a function defined over the domain A, and we have to find the unknown f for points in A.

The possible functions can often be restricted to functions whose square has finite integration over A. This space is usually called $L_2(A)$ space where the scalar product is defined as:

$$\langle u, v\rangle = \int_A u(p)v(p)\,dA \tag{2}$$

An operator $\mathcal{L}$ is said to be **symmetric** if for any u, v in $L_2(A)$:

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}v\rangle. \tag{3}$$

Furthermore, an operator $\mathcal{L}$ is **positive** if it is symmetric and for any u in $L_2(A)$:

$$\langle \mathcal{L}u, u\rangle \;\geq 0 \qquad \wedge \qquad \langle \mathcal{L}u, u\rangle = 0 \quad \text{if and only if } u = 0. \tag{4}$$

If operator $\mathcal{L}$ is linear, symmetric and positive, then according to the *minimal theorem of quadratic functionals* [Ode76] the solution of equation $\mathcal{L}f(p) = g(p)$ can also be found as the stationary point of the following functional:

$$I(f) = \langle \mathcal{L}f, f\rangle - 2\langle g, f\rangle + \langle g, g\rangle \tag{5}$$

Note that $\langle g, g\rangle$ makes no difference in the stationary point, since it does not depend on f, but it simplifies the resulting formula.

Let us suppose that a function f_0 satisfies the integral equation, that is:

$$\mathcal{L}f_0 = g \tag{6}$$

In order to prove that if functional $I(f)$ is extreme for some function f then $f = f_0$ must hold, a sequence of identity relations based on the assumption that $\mathcal{L}$ is linear, symmetric and positive can be used:

$$\begin{gathered} I(f) = \langle \mathcal{L}f, f\rangle - 2\langle g, f\rangle + \langle g, g\rangle = \langle \mathcal{L}f, f\rangle - 2\langle \mathcal{L}f_0, f\rangle + \langle g, g\rangle = \\ \langle \mathcal{L}f, f\rangle - \langle \mathcal{L}f_0, f\rangle - \langle f_0, \mathcal{L}f\rangle + \langle g, g\rangle = \\ \langle \mathcal{L}f, f\rangle - \langle \mathcal{L}f_0, f\rangle - \langle \mathcal{L}f, f_0\rangle + \langle \mathcal{L}f_0, f_0\rangle - \langle \mathcal{L}f_0, f_0\rangle + \langle g, g\rangle = \\ \langle \mathcal{L}(f - f_0), (f - f_0)\rangle - \langle \mathcal{L}f_0, f_0\rangle + \langle g, g\rangle \end{gathered} \tag{7}$$

Since only the term $\langle \mathcal{L}(f - f_0), (f - f_0)\rangle$ depends on f and this term is minimal if and only if $f - f_0$ is zero due to the assumption that $\mathcal{L}$ is positive, the functional is really extreme for that f which satisfies equation 1.

Hence instead of finding a function that satisfies $\mathcal{L}f(p) = g(p)$, we can determine the same function by finding that f which minimizes functional $I(f)$.

Radiosity calculation

Suppose that the radiosity should be calculated on surface A (figure 1).

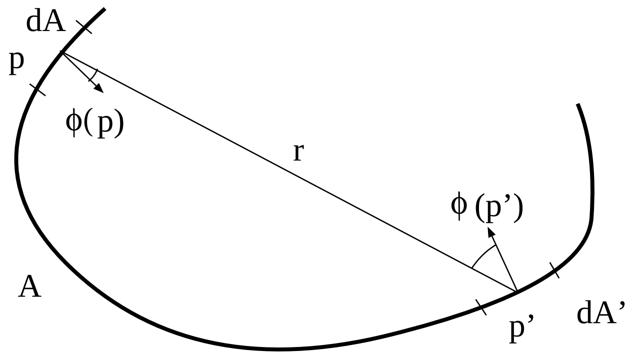

Figure 1: Geometry of radiosity calculation

Let the diffuse coefficient be $\varrho(p)$ at point p and the visibility indicator between points p and p' be $H(p,p')$. Using the notations of figure 1, and denoting the radiosity and emission at point p by $B(p)$ and $E(p)$ respectively, the basic radiosity equation is:

$$B(p)\ dA = E(p)\ dA + \varrho(p) \int_A B(p')\ f(p,p')\ dA' \cdot dA \tag{8}$$

where $f(p,p')$ is the **point-to-point form factor**:

$$f(p,p') = H(p,p')\ \frac{\cos\phi(p)\cdot\cos\phi(p')}{r^2\pi} \tag{9}$$

Dividing both sides by dA, the **radiosity equation** is then:

$$B(p) = E(p) + \varrho(p) \int_A B(p')\ f(p,p')\ dA' \tag{10}$$

Let us define a linear operator $\mathcal{L}$ as:

$$\mathcal{L}B(p) = B(p) - \varrho(p) \int_A B(p')\ f(p,p')\ dA' \tag{11}$$

Then the radiosity equation can also be written as follows:

$$\mathcal{L}B(p) = E(p) \tag{12}$$

The solution of the radiosity problem means to find a function B satisfying this equation. Since we are interested in finite energy solutions only, the search space of possible functions can obviously be restricted to functions whose square has finite integration over A, that is, which are in $L_2(A)$.

Unfortunately $\mathcal{L}$ is not symmetric in its original form due to the asymmetry of the radiosity equation that depends on only $\varrho(p)$ but not on $\varrho(p')$. One possible approach to this problem is the subdivision of surfaces into finite patches having constant diffuse coefficients, and working with multi-variate functionals, but this results in a significant computational overhead [SK93].

Now another solution is proposed that eliminates the asymmetry by calculating $B(p)$ indirectly through the generation of $B(p)/\sqrt{\varrho(p)}$. In order to do this, both sides of the radiosity equation are divided by $\sqrt{\varrho(p)}$:

$$\frac{E(p)}{\sqrt{\varrho(p)}} = \frac{B(p)}{\sqrt{\varrho(p)}} - \sqrt{\varrho(p)} \int_A \frac{B(p')}{\sqrt{\varrho(p')}} \sqrt{\varrho(p')}\ f(p,p')\ dA'. \tag{13}$$

Let us define $B^*(p)$, $E^*(p)$ and $g(p,p')$ by the following formulae:

$$B^*(p) = \frac{B(p)}{\sqrt{\varrho(p)}}, \qquad E^*(p) = \frac{E(p)}{\sqrt{\varrho(p)}}, \qquad g(p,p') = f(p,p')\sqrt{\varrho(p)\varrho(p')} \tag{14}$$

Using these definitions, we get the following form of the original radiosity equation:

$$E^*(p) = B^*(p) - \int_A B^*(p')\ g(p,p')\ dA' \tag{15}$$

This integral equation is defined by a symmetric, positive linear operator $\mathcal{L}^*$:

$$\mathcal{L}^* B^*(p) = B^*(p) - \int_A B^*(p')\ g(p,p')\ dA' \tag{16}$$

The symmetry is proven as follows:

$$\langle \mathcal{L}^* u, v \rangle = \int_A u(p)v(p)\ dA - \int_A \int_A v(p)u(p')\ g(p,p')\ dA'dA =$$

$$\int_A v(p)u(p)\ dA - \int_A \int_A u(p)v(p')\ g(p',p)\ dA'dA = \langle v, \mathcal{L}^* u \rangle \tag{17}$$

since $g(p,p') = g(p',p)$.

To prove the positive property, the physical meaning of the point-to-point form factor is taken into consideration. The radiosity equation tells us that if the surface has $B(p)$ radiosity distribution, then the energy radiated onto an elemental surface dA at p is:

$$dA \cdot \int_A B(p') f(p,p')\ dA' \tag{18}$$

Thus the total energy radiated onto any part of the surface A is:

$$\int_A \int_A B(p') f(p,p')\ dA'dA \tag{19}$$

This certainly cannot exceed the total energy radiated by the surface, thus:

$$\int_A \int_A B(p') f(p,p')\ dA'dA \leq \int_A B(p)\ dA \tag{20}$$

Since $g(p,p') = f(p,p')\sqrt{\varrho(p)\varrho(p')}$ and for physically correct models the diffuse coefficients are less than 1, we also get:

$$\int_A \int_A B(p') g(p,p')\ dA'dA \leq \int_A B(p)\ dA \tag{21}$$

This equation is true for any kind of radiosity distribution, thus it is also valid for any non-negative function B.

Now let us turn to the definition of positiveness of an operator and apply some basic inequalities:

$$\langle \mathcal{L}^* u, u \rangle = \int_A u^2(p)\ dA - \int_A \int_A u(p')u(p)g(p,p')\ dA'dA \ \geq$$

$$\int_A u^2(p)\ dA - \int_A \int_A |u(p')\sqrt{g(p,p')}| \cdot |u(p)\sqrt{g(p,p')}|\ dA'dA \tag{22}$$

According to the Cauchy-Schwarz inequality we can have the following upper bound for the second term:

$$\int_A \int_A |u(p')\sqrt{g(p,p')}| \cdot |u(p)\sqrt{g(p,p')}| \; dA'dA \leq$$

$$\leq \sqrt{\int_A \int_A u^2(p')g(p,p') \; dA'dA} \cdot \sqrt{\int_A \int_A u^2(p)g(p,p') \; dAdA'} =$$

$$\int_A \int_A u^2(p')g(p,p') \; dA'dA, \tag{23}$$

since $g(p,p') = g(p',p)$.

Since $u^2(p)$ is obviously a positive function inequality 21 can be applied, thus:

$$\int_A \int_A u^2(p')g(p,p') \; dA'dA \leq \int_A u^2(p) \; dA, \tag{24}$$

Inserting equations 23 and 24 into equation 22, we get:

$$\langle \mathcal{L}^* u, u \rangle = \int_A u^2(p) \; dA - \int_A \int_A u(p')u(p)g(p,p') \; dA'dA \geq 0 \tag{25}$$

We can see that $\langle \mathcal{L}^* u, u \rangle$ is 0 if $u = 0$, and examining the inequalities used we can also come to the conclusion that if $u \neq 0$, then $\langle \mathcal{L}^* u, u \rangle > 0$ holds. Hence $\mathcal{L}^*$ is really positive.

This means that the solution of the radiosity equation 15 is equivalent to finding the stationary point of the following functional:

$$I(B^*) = \langle \mathcal{L}^* B^*, B^* \rangle - 2\langle E^*, B^* \rangle + \langle E^*, E^* \rangle =$$

$$\int_A (E^*(p) - B^*(p))^2 dA - \int_A \int_A B^*(p) \; B^*(p') \; g(p,p') \; dAdA' \tag{26}$$

This extreme property of functional I can also be proven by generating the functional's first variation and making it equal to zero:

$$\delta I = \frac{\partial I(B^* + \alpha \; \delta B)}{\partial \alpha}|_{\alpha=0} = 0 \tag{27}$$

Using elementary derivation rules and taking into account the following symmetry relation:

$$\int_A \int_A B^*(p) \; \delta B(p') \; g(p,p') \; dAdA' = \int_A \int_A \delta B(p) \; B^*(p') \; g(p,p') \; dAdA', \tag{28}$$

the formula of the first variation is transformed to:

$$\delta I = \int_A [E^*(p) - B^*(p) + \int_A B^*(p') \cdot g(p,p') \; dA'] \cdot \delta B \; dA = 0 \tag{29}$$

The term closed in brackets should be zero to make the expression zero for any δB variation. That is exactly the original radiosity equation, hence finding the stationary point of functional I is really equivalent to solving integral equation 15.

Finding the extremum of the functional

In order to find the extremum of functional $I(B^*)$, Ritz's method is used. Assume that the unknown function B^* is approximated by a function series:

$$B^*(p) \approx \sum_{k=1}^{n} a_k \cdot b_k(p) \tag{30}$$

where $(b_1, b_2, ..., b_n)$ form a complete function system (that is, any piecewise continuous function can be approximated by their linear combination), and $(a_1, a_2, ..., a_n)$ are unknown coefficients. This assumption makes functional $I(B^*)$ an n-variate function $I(a_1, ..., a_n)$, which is extreme if all the partial derivatives are zero. Having made every $\partial I / \partial a_k$ equal to zero, a linear equation system can be derived for the unknown a_k-s $(k = \{1, 2, ..., n\})$:

$$\sum_{i=0}^{n} a_i [\int_A b_i(p) b_k(p) dA - \int_A \int_A b_k(p) b_i(p') g(p, p') \; dA dA'] = \int_A E^*(p) b_k(p) dA \tag{31}$$

This general formula provides a linear equation for any kind of complete function system $b_1, ..., b_n$, thus it can be regarded as a basis of many different radiosity approximation techniques, because the different selection of basis functions, b_i, results in different methods of determining the radiosity distribution.

In this paper three types of function bases are discussed:

- piecewise constant functions which lead to the traditional method, proving that the original approach is a special case of this general framework.
- piecewise linear functions which, as we shall see, are not more difficult than the piecewise constant approximations, but they can provide more accurate solutions.
- harmonic (cosine) functions where the basis functions are not of finite element type because they can approximate the radiosity distribution everywhere not just in a restricted part of the domain, and thus fall into the category of global element methods.

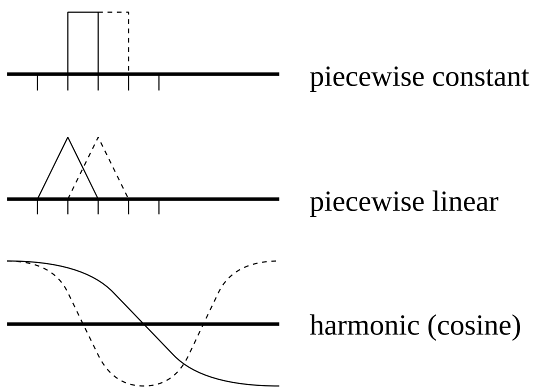

Figure 2: 1-dimensional analogies of proposed basis functions

Piecewise constant radiosity approximation

Following the original method, an appropriate set of basis functions b_k can be defined having broken down the surface into surface elements ΔA_1, ΔA_2,...,ΔA_n, and assuming:

$$b_k(p) = \begin{cases} 1 \text{ if } p \text{ is on } \Delta A_k \\ \\ 0 \text{ otherwise} \end{cases} \tag{32}$$

If the emission E and the diffuse coefficient ϱ are assumed to be constant on the elemental surface ΔA_k and equal to E_k and ϱ_k respectively, equation 31 will have the following form:

$$a_k \Delta A_k - \sum_{i=0}^{n} a_i [\int_{\Delta A_k} \int_{\Delta A_i} g(p, p')\ dA dA'] = \frac{E_k}{\sqrt{\varrho_k}} \Delta A_k \tag{33}$$

According to the definition of basis function b_k, the radiosity of patch k is:

$$B_k = B_k^* \sqrt{\varrho_k} = a_k \sqrt{\varrho_k} \tag{34}$$

Substituting this into equation 33 and using the definition $g(p, p') = f(p, p')\sqrt{\varrho(p)\varrho(p')}$, we get:

$$B_k \Delta A_k - \varrho_k \sum_{i=0}^{n} B_i [\int_{\Delta A_k} \int_{\Delta A_i} f(p, p')\ dA dA'] = E_k \Delta A_k \tag{35}$$

Let us introduce the **patch-to-patch form factor** as follows:

$$F_{ki} = \frac{1}{\Delta A_k} \int_{\Delta A_k} \int_{\Delta A_i} f(p, p')\ dA dA' \tag{36}$$

Note that this is the usual definition taking into account the interpretation of $f(p, p')$ in equation 9. Dividing both sides by ΔA_k, the linear equation is then:

$$B_k - \varrho_k \sum_{i=0}^{n} B_i F_{ki} = E_k \tag{37}$$

This is exactly the well known linear equation of original radiosity method [GCT86].

Now let us begin to discuss how to define and use other, more effective function bases.

Linear finite element techniques

Let us decompose the surface into planar triangles and assume that the radiosity variation is linear on these triangles. Thus, each vertex i of the triangle mesh will correspond to a "tent shaped" basis function b_i that is 1 at this vertex and linearly decreases to 0 on the triangles incident to this vertex.

Placing the center of the coordinate system into vertex i, the position vector of points on an incident triangle can be expressed by a linear combination of the edge vectors $\vec{x}, \vec{y}$:

$$p = \alpha \vec{x} + \beta \vec{y} \tag{38}$$

with $\alpha, \beta \geq 0 \quad \wedge \quad \alpha + \beta \leq 1$.

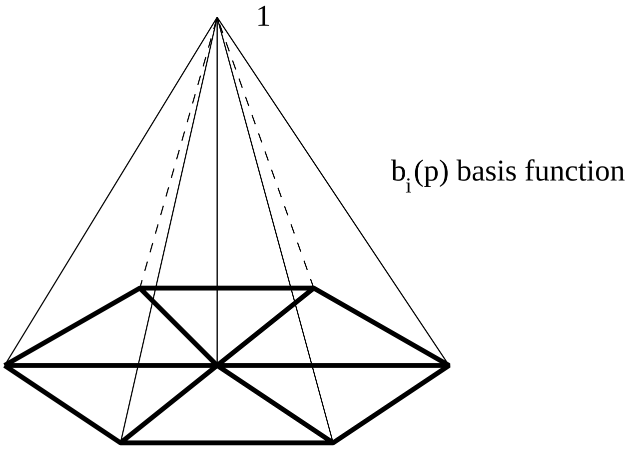

Figure 3: Linear basis function in 3-dimension

Thus, the surface integral of some function F on a triangle can be written as follows:

$$\int_{\Delta A} F(p)dA = \int_{\beta=0}^{1} \int_{\alpha=0}^{1-\beta} F(\alpha,\beta)|\vec{x}\times\vec{y}|d\alpha d\beta = 2\Delta A \int_{\beta=0}^{1} \int_{\alpha=0}^{1-\beta} F(\alpha,\beta)\ d\alpha d\beta \tag{39}$$

If $F(\alpha,\beta)$ is a polynomial function, then its surface integration can be determined in closed form by this formula.

The basis function which is linearly decreasing on the triangles can be conveniently expressed by coordinates α,β:

$$b_i(\alpha,\beta) = 1-\alpha-\beta \tag{40}$$
$$b_{i'}(\alpha,\beta) = \alpha \tag{41}$$
$$b_{i''}(\alpha,\beta) = \beta \tag{42}$$
$$b_j = 0 \qquad \text{if } j \neq i, i', i'' \tag{43}$$

where i, i' and i'' are the three vertices of the triangle.

Let us consider the general equation (equation 31) defining the weights of basis functions; that is the radiosities at triangle vertices for linear finite elements. Although its integrals can be evaluated directly, it is worth examining whether further simplification is possible. Equation 31 can also be written as follows:

$$\int_{A} [\sum_{i=0}^{n} a_i\{b_i(p) - \int_{A} b_i(p')g(p,p')\ dA'\} - E^*(p)] \cdot b_k(p)\ dA = 0 \tag{44}$$

The term enclosed in brackets is a piecewise linear expression according to our assumption if E^* is also linear. The integration of the product of this expression and any linear basis function is zero. That is possible if the term in brackets is constantly zero, thus an equivalent system of linear equation can be derived by requiring the closed term to be zero in each vertex k, $k = \{1,2,...n\}$ (this implies that the function will be zero everywhere because of linearity):

$$a_k - \sum_{i=0}^{n} a_i \int_{A} b_i(p')g(p_k,p')\ dA' = E^*_k \tag{45}$$

As in the case of piecewise constant approximation, the diffuse coefficient $\varrho(p)$ is assumed to be equal to ϱ_k at vertex k, and using the definitions of the normalized radiosities and the linear basis functions we can conclude that:

$$a_k = B_k^* = \frac{B_k}{\sqrt{\varrho_k}}, \qquad E_k^* = \frac{E_k}{\sqrt{\varrho_k}} \tag{46}$$

Substituting this into equation 45, we get:

$$B_k - \varrho_k \sum_{i=0}^{n} B_i [\int_{\Delta A_i} b_i(p') f(p_k, p') \sqrt{\frac{\varrho(p')}{\varrho(p_i)}}\, dA'] = E_k \tag{47}$$

Let us introduce the **vertex-patch form factor** P_{ki}:

$$P_{ki} = \int_{A} b_i(p') f(p_k, p') \sqrt{\frac{\varrho(p')}{\varrho(p_i)}}\, dA' \tag{48}$$

If the diffuse coefficient can be assumed to be (approximately) constant on the triangles adjacent to vertex i, then:

$$P_{ki} \approx \int_{A} b_i(p') f(p_k, p')\, dA' \tag{49}$$

The linear equation of the vertex radiosities is then:

$$B_k - \varrho_k \sum_{i=0}^{n} B_i P_{ki} = E_k \tag{50}$$

This is almost the same as the linear equation describing the piecewise constant approximation (equation 37), except that:

- Unknown parameters $B_1, ..., B_k$ represent now vertex radiosities rather than patch radiosities. According to Euler's law, the number of vertices of a triangular faced polyhedron is half of the number of its faces plus two. Thus the size of the linear equation is almost the same as for the number of quadrilaterals used in the original method.

- There is no need for double integration and thus the linear approximation requires a simpler numerical integration to calculate the form factors than constant approximation.

The vertex-patch form factor can be evaluated by the techniques developed for patch-to-patch form factors taking account also the linear variation due to b_i. This integration can be avoided, however, if linear approximation of $f(p_k, p')$ is acceptable. One way of achieving this is to select the subdivision criterion of surfaces into triangles accordingly.

A linear approximation can be based on point-to-point form factors between vertex k and the vertices of triangle $\Delta A'$. Let the $f(p_k, p)$ values of the possible combinations of point p_k and the vertices be F_1, F_2, F_3 respectively. A linear interpolation of the point-to-point form factor between p_k and $p' = \alpha' \vec{x}' + \beta' \vec{y}'$ is:

$$f(p_k, p') = \alpha' F_1 + \beta' F_2 + (1 - \alpha' - \beta') F_3 \tag{51}$$

Using this assumption the surface integral defining P_{ki} can be expressed in closed form.

Global element approach - harmonic functions

In contrast to previous cases, the application of harmonic functions does not require the subdivision of surfaces into planar polygons, but deals with the original geometry. This property makes it especially useful when the view-dependent rendering phase uses ray-tracing.

Suppose surface A is defined parametrically by a position vector function, $\vec{r}(u,v)$, where parameters u and v are in the range of $[0,1]$.

Let a representative of the basis functions be:

$$b_{ij} = \cos(i\pi u) \cdot \cos(j\pi v) = C_u^i C_v^j \tag{52}$$

(C_u^i substitutes $\cos(i\pi u)$ for notational simplicity). Note that the basis functions have two indices, hence the sums should also be replaced by double summation in equation 31. Examining the basis functions carefully, we can see that the goal is the calculation of the Fourier series of the radiosity distribution.

In contrast to the finite element method, the basis functions are now non-zero almost everywhere in the domain, so they can approximate the radiosity distribution in a wider range. For that reason, approaches applying this kind of basis function are called **global element methods**.

In the radiosity method the most time consuming step is the evaluation of the integrals appearing as coefficients of the linear equation system (equation 31). By the application of cosine functions, however, the computational time can be reduced significantly, because of the orthogonal properties of the trigonometric functions, and also by taking advantage of effective algorithms, such as Fast Fourier Transform (FFT).

In order to illustrate the idea, the calculation of

$$\int_A E^*(p) b_{kl}(p)\ dA$$

for each k,l is discussed. Since $E^*(p) = E^*(\vec{r}(u,v))$, it can be regarded as a function defined over the square $[0,1]^2$. Using the equalities of surface integrals, and introducing the notation $J(u,v) = |\partial\vec{r}/\partial u \times \partial\vec{r}/\partial v|$ for surface element magnification, we get:

$$\int_A E^*(p) b_{kl}(p)\ dA = \int_0^1 \int_0^1 E^*(\vec{r}(u,v)) b_{kl}(u,v) J(u,v)\ dudv \tag{53}$$

Let us mirror the function $E^*(\vec{r}) \cdot J(u,v)$ onto coordinate system axes u and v, and repeat the resulting function having its domain in $[-1,1]^2$ infinitely in both directions with period 2. Due to mirroring and periodic repetition, the final function $\hat{E}(u,v)$ will be even and periodic with period 2 in both directions. According to the theory of the Fourier series, the function can be approximated by the following sum:

$$\hat{E}(u,v) \approx \sum_{i=0}^{m} \sum_{j=0}^{m} E_{ij} C_u^i C_v^j \tag{54}$$

All the Fourier coefficients E_{ij} can be calculated by a single, 2-dimensional FFT. (A D-dimensional FFT of N samples can be computed by taking DN^{D-1} number of 1-dimensional FFTs [Nus82] [PFTV88].) Since $\hat{E}(u,v) = E^*(\vec{r}) \cdot J(u,v)$ if $0 \le u,v \le 1$, this Fourier series and the definition of the basis functions can be applied to equation 53, resulting in:

$$\int_A E^*(p) b_{kl}(p)\ dA = \int_{u=0}^{1} \int_{v=0}^{1} \sum_{i=0}^{m} \sum_{j=0}^{m} E_{ij} C_u^i C_v^j b_{kl}(u,v)\ dudv =$$

$$\sum_{i=0}^{m} \sum_{j=0}^{m} E_{ij} \int_0^1 C_u^i C_u^k du \int_0^1 C_v^j C_v^l dv = \begin{cases} E_{0,0} & \text{if } k=0 \text{ and } l=0 \\ E_{0,l}/2 & \text{if } k=0 \text{ and } l \neq 0 \\ E_{k,0}/2 & \text{if } k \neq 0 \text{ and } l=0 \\ E_{k,l}/4 & \text{if } k \neq 0 \text{ and } l \neq 0 \end{cases} \qquad (55)$$

Consequently, the integral can be calculated in closed form, having replaced the original function by Fourier series. Similar methods can be used to evaluate the other integrals. In order to compute

$$\int_A b_{ij}(p) b_{kl}(p) dA$$

$J(u,v)$ must be Fast Fourier Transformed.

To calculate

$$\int_A \int_A b_k(p) b_i'(p) g(p,p')\ dA dA'$$

the Fourier transform of

$$g(p(u,v), p'(u',v')) \cdot J(u,v) J(u',v')$$

is needed. Unfortunately the latter requires a 4D FFT which involves many operations. Nevertheless, this transform can be realized by two 2-dimensional FFTs if $g(p,p')$ can be assumed to be nearly independent of either p or p', or it can be approximated by a product form of p and p' independent functions.

Conclusions

In this paper, a general methodology has been proposed to derive new radiosity solutions. Three special function bases were discussed to demonstrate the ideas and also the merits of this approach. The third one resulted in a global element method that uses basis functions which are capable of approximating the desired function everywhere in its domain and does not require the subdivision of the original surfaces into planar polygons. Global element approaches offer exciting alternatives of finite element methods. As has been demonstrated, they are more complicated theoretically, but allow for effective techniques. It should be mentioned that other global function bases can also be useful. As for example, Haar wavelet functions and Chebyshev polynomials are effective in approximation, and similar techniques to FFT can be developed for their computation.

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