A Brief Survey of Aesthetic Curves

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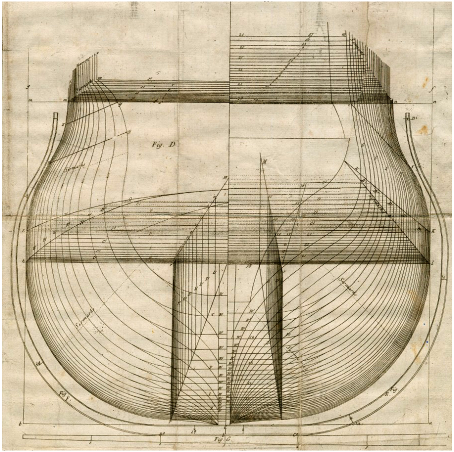
Outline

- Introduction
  - Motivation
  - Classic aesthetic curves
- Modern aesthetic curves
  - Class A Bézier curves
  - Log-aesthetic curves
  - Extensions of LA curves
- Conclusion

Source: Edson International.



Source: William Sutherland, *The Shipbuilders Assistant*.

Fair curves

Definition

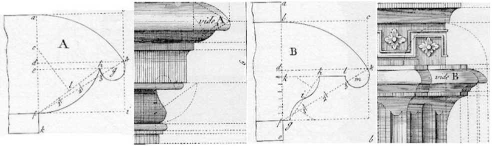
A curve is fair if its curvature plot is continuous and consists of only a few monotone pieces. [Fa01]

- Generic curves require smoothing
  - by post-processing
  - by variational fitting techniques
- Curve representations with intrinsic smoothness?
  - Curves with monotone curvature plots
  - Limit our scope to 2D curves
- Cesàro equation
  - Curvature as a function of arc length

[Fa01] G. Farin, *Curves and Surfaces for CAGD*, 5th Ed., Morgan Kaufmann, 2001.

Circle

- Loved since the beginning of time
- Most basic curve
- Cesàro equation: $\kappa(s) = c$ (const)
- Prevalent in CAD (and everywhere)
- Its use is limited in itself
  - Combination of circular arcs and straight line segments
  - Only C^0 or G^1 continuity



Source: Batty Langley, *Gothic Architecture*.

Neolithic cult symbol, Spain.

Parabola

- Menaechmus (4th century BC)
- CAD – quadratic Bézier curve
  - TrueType fonts
  - Macromedia Flash
- Not always monotone curvature ★
- Nice physical properties
  - Used in bridges, arches
  - Also in antennas, reflectors

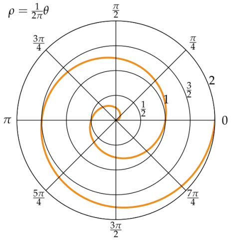
Apollonius' *Conics* (in arabic, 9th c).

The Golden Gate bridge.

A parabola antenna.

Archimedes' (arithmetic) spiral

- Archimedes (3rd century BC)
  - Used for squaring the circle
- A line rotates with const. ω, and a point slides on it with const. v
- Polar equation: $r = a + b\theta$

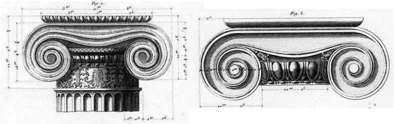


Source: Wikipedia (Archimedean spiral).

Great Mosque of Samarra, Iraq.

Archimedean spiral

- Generalized Archimedes' spiral
- Polar equation: $r = a + b\theta^{1/n}$
- $n = -2 \Rightarrow$ lituus (Cotes, 18th c.)
  - Augur's curved staff
  - Frequently used for volutes
- $n = -1 \Rightarrow$ hyperbolic spiral
- $n = 2 \Rightarrow$ Fermat's spiral



Volutes (Source: Julien David LeRoy,
Les ruines plus beaux des monuments de la Grèce).

Crosier of Archbishop
Heinrich of Finstingen.

Circle involute

- Huygens (17th century)
  - Used for pendulum clocks
- Similar to Archimedes' spiral
  - Often confused, but not the same
  - This one has parallel curves
- Traced by the end of a rope coiled on a circular object ★
- Cesàro equation: $\kappa(s) = c/\sqrt{s}$
- Used for cog profiles (since Euler) and scroll compressors (pumps)
- Occurs frequently in nature and art

Hawaiian fern

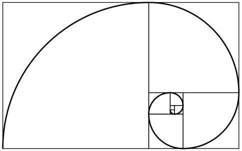
Coiled millipede.

Logarithmic spiral

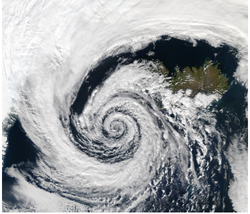
- Descartes & Bernoulli (17th c.)
  - "spira mirabilis"
- The golden spiral is also logarithmic
- Polar equation: $r = ae^{b\theta}$
- Cesàro equation: $\kappa(s) = c/s$
- Very natural, self-similar pattern
  - Shells, sunflowers, cyclones etc.

Lower part of Bernoulli's gravestone

(but the spiral is Archimedean).



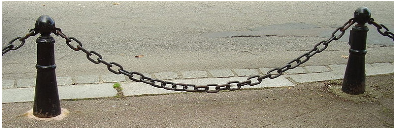
Fibonacci spiral, approximating the golden spiral (Wikipedia).



Cyclone over Iceland (NASA).

Catenary curves

- Hooke; Leibniz, Huygens & Bernoulli (17th century)
  - Curve of a hanging chain or cable
- Equation: $y = a\cosh(x/a)$
- Cesàro equation: $\kappa(s) = a/(s^2 + a^2)$
- Used in architecture
  - Design of bridges / arches



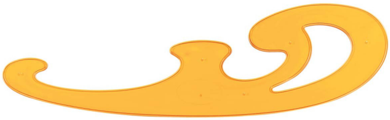
A hanging chain showing a catenary curve.

Gaudi's design of a church

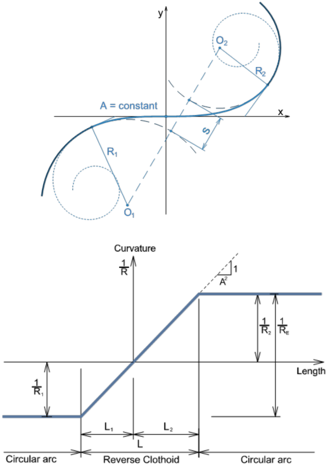
at Santa Coloma de Cervello.

Clothoid (Euler/Cornu spiral)

- Euler (18th c.) & Cornu (19th c.)
- G^2 transition between a circular arc and a straight line
- Cesàro equation: $\kappa(s) = c \cdot s$
- French curves have clothoid edges
- Used in urban planning
  - Railroad / highway design
  - Linear centripetal acceleration



A french curve.



Source: PWayBlog.

Definition [ML98, Fa06]

- Bézier curves are *typical*, if each "leg" of the control polygon is obtained by the same rotation and scale of the previous one:

$$\Delta P_{i+1} = s \cdot R \Delta P_i \quad [\Delta P_j = P_{j+1} - P_j]$$

where s is the scale factor, R is a rotation matrix by α

- Class A Bézier curves are more general:

$$\Delta P_i = M^i v$$

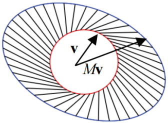
where M is a 2×2 matrix and v is a unit vector

- These curves can be extended to 3D, as well

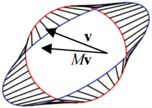
[ML98] Y. Mineur, T. Lichah, J. M. Castelain, H. Giaume,
A shape controled fitting method for Bézier curves. In: CAGD 15(9), pp. 879–891, 1998.
[Fa06] G. Farin, *Class A Bézier curves*. In: CAGD 23(7), pp. 573–581, 2006.

Properties

- Goal: continuous & monotone curvature
- Typical curves need constraints on s and α
  - $\cos\alpha > 1/s$ (if $s > 1$) or $\cos\alpha > s$ (if $s \leq 1$)
- Class A Bézier curves need constraints on M
- Constraints [Fa06]: the segments $v - Mv$ do not intersect the unit circle for any unit vector v
- Constraints [CW08]:
  - $M = SD^iS^{-1}$, where S is orthogonal, D is diagonal (assuming a symmetric M)
  - $d_{11} \geq 1$, $d_{22} \geq 1$, $2d_{11} \geq d_{22} + 1$, $2d_{22} \geq d_{11} + 1$
- Similar constraints for 3D curves



Matrix satisfying the constraints.

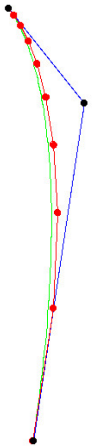


Matrix not satisfying the constraints.

[CW08] J. Cao, G. Wang, *A note on Class A Bézier curves*. In: CAGD 25(7), pp. 523–528, 2008.

Interpolation

- [Fa06]: Input: P_0, v and the end tangent v_n
  - Rotation angle $\alpha = \angle(v, v_n)/n$
  - Scale factor $s = (\|v_n\|/\|v\|)^{1/n}$
  - Condition: $\cos\alpha > 1/s$
    $\Rightarrow$ true if n is large enough
- Problems:
  - Cannot set the end position
    $\Rightarrow$ not designer-friendly
  - For $\|v_n\| \approx \|v\|$ the degree n must be very high
- [YS08]: Input: position and tangent at both ends
  - Using 3 control points a_0, a_1, a_2



[YS08] N. Yoshida, T. Saito, *Interactive Control of Planar Class A Bézier Curves using Logarithmic Curvature Graphs*. In: CAD&A 5(1-4), pp. 121–130, 2008.

Three-point interpolation ★

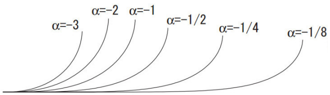
- Needed: P_0, α, v, s (assume fixed n)
- $P_0 = a_0$
- $\alpha = \angle(a_1 - a_0, a_2 - a_1)/n$
- $v = b_0 \cdot \frac{a_1 - a_0}{\|a_1 - a_0\|} =: b_0 \cdot u$
- b_0 is defined by the equation

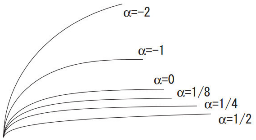
$$\sum_{j=0}^{n-1} b_0 M^j u = a_2 - a_0, \quad M = s \cdot R(\alpha)$$

- For $n = 3$ this is a quadratic equation,
  otherwise polynomial root finding algorithms are needed
  $\Rightarrow$ just approximates the endpoint
- For large n these curves converge to logarithmic spirals

Generalization of classic curves [Mi06]

- Curve defined by its curvature plot
- Cesàro equation:
  $\kappa(s) = (c_0 \cdot s + c_1)^{-1/\alpha}$
- $c_1 = 0,\ \alpha = 2 \Rightarrow$ circle involute
- $c_1 = 0,\ \alpha = 1 \Rightarrow$ logarithmic spiral
- $c_1 = 0,\ \alpha = 0 \Rightarrow$ Nielsen's spiral
  - Plot of $(a \cdot \mathrm{ci}(t), a \cdot \mathrm{si}(t))$
  - ci / si : cos/sin integral functions
  - $\kappa(s) = e^{s/a}/a$ [YS08]
- $c_1 = 0,\ \alpha = -1 \Rightarrow$ clothoid





[Mi06] K. T. Miura, *A general equation of aesthetic curves and its self-affinity*. In: CAD&A 3(1-4), pp. 457–464, 2006.

Curve equation

- Represented as a complex function:

$$C(s) = P_0 + \int_0^s e^{i\theta} \mathrm{d}s$$

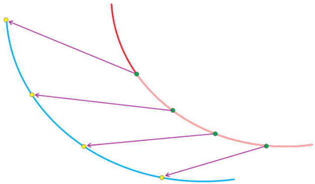
where the tangent angle θ is

$$\theta = \frac{\alpha(c_0 s + c_1)^{1-\frac{1}{\alpha}}}{(\alpha - 1)c_0} + c_2$$

- Note that $e^{i\theta} = (\cos\theta + i\sin\theta)$,
  and that in arc-length parameterization
  - the first derivative is a unit vector
  - curvature is the centripetal acceleration
    $\Rightarrow$ its integral is the angular velocity

Properties

- Self-affinity
  - Weaker than self-similarity
  - The "tail" of a log-aesthetic arc can be affinely transformed into the whole curve
- Natural shape
  - Egg contour, butterfly wings, etc.
- Also appears in art and design
  - Japanese swords, car bodies, etc.



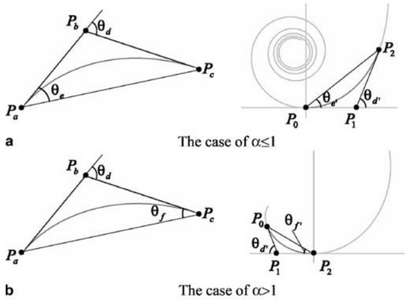
Scaling a segment shows self-affinity.

A japanese sword.

A swallowtail butterfly.

Interpolation [YS06] ★

- Input: 3 control points P_a, P_b, P_c (as before), α fixed
- Idea: find a segment of the curve in standard form
  - $P_0 = 0,\ \theta(0) = 0,\ \kappa(0) = 1$
  - Transform the control points to match a segment



[YS06] N. Yoshida, T. Saito, *Interactive aesthetic curve segments*.
In: The Visual Computer 22(9-11), pp. 896–905, 2006.

Interpolation (2)

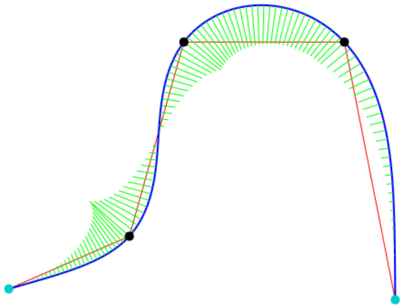
- In this form, the curve is defined by a scalar Λ:
  - $c_0 = \alpha\Lambda$, $c_1 = 1$, $c_2 = \frac{1}{(\alpha-1)\Lambda}$
- P_0 is the origin, P_2 is $C(s_0)$ where s_0 is the total length
  - s_0 can be computed from θ_d
- P_1 is found by intersection:

$$P_1 = \mathrm{Re}\left[P_2 + e^{i\theta_d} \cdot \left(-\frac{\mathrm{Im}(P_2)}{\mathrm{Im}(e^{i\theta_d})}\right)\right]$$

- The input triangle and transformed triangle should be similar
  - Find the value of Λ by iterative bisection
  - For $\alpha = 1$, Λ can be arbitrarily large (open-ended bisection)
  - Otherwise $\Lambda \in [0, \theta_d/(1-\alpha)]$
- Quite a few corner cases...

Discrete spline interpolation [SK00] ★

- Input:
  - Points to interpolate
- Output:
  - Discrete curve (polygon)
  - Open or closed
  - Input points are knots (segment boundaries)
  - Each segment is LA, connected with G^2
- Originally for clothoids, but easily adapted to LAC



[SK00] R. Schneider, L. Kobbelt, *Discrete fairing of curves and surfaces based on linear curvature distribution*. Max Planck Institut für Informatik, Saarbrücken, 2000.

Discrete spline interpolation (2) – Algorithm

1. Subsample the input $\rightarrow Q_i^0$
2. Compute discrete curvatures at input points:

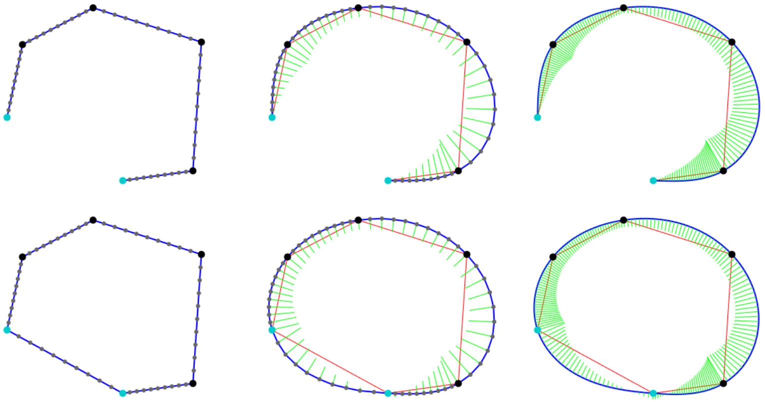
$$\kappa_i = 2\frac{\det(Q_i^k - Q_{i-1}^k, Q_{i+1}^k - Q_i^k)}{\left\|Q_i^k - Q_{i-1}^k\right\| \left\|Q_{i+1}^k - Q_i^k\right\| \left\|Q_{i+1}^k - Q_{i-1}^k\right\|}$$

3. Assign target curvatures to other points (based on α)
4. Compute the new position of the other points
   - 4.1 Local discrete curvature equals target curvature
   - 4.2 Segments are arc-length parameterized:

$$\left\|Q_i^{k+1} - Q_{i-1}^k\right\| = \left\|Q_{i+1}^k - Q_i^{k+1}\right\|$$

5. Back to step 2 (unless change was $< \varepsilon$ or too many iterations)

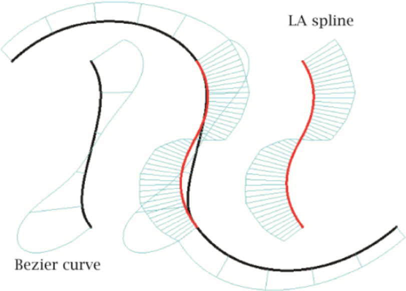
Discrete spline interpolation (3) – Example

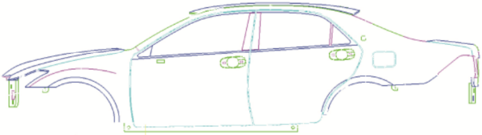


(a) Subsampled input. (b) Output curve. (c) Dense output curve.

G^2 LA spline [MS13]

- 3-segment spline, connecting with G^2 continuity
- Input: position, tangent & curvature at the endpoints
- Iterative; uses a Bézier curve to estimate total arc length
- Capable of S-shapes





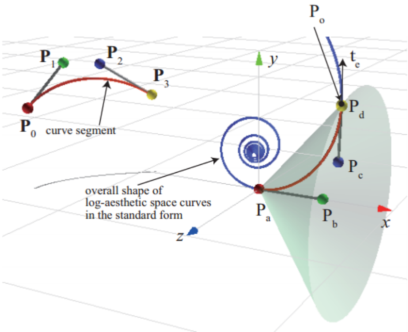
[MS13] K. T. Miura, D. Shibuya, R. U. Gobithaasan, Sh. Usuki, *Designing Log-aesthetic Splines with G^2 Continuity*. In: CAD&A 10(6), pp. 1021–1032, 2013.

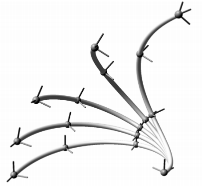
LA space curve [MF06, YF09]

- Extension is based on an additional equation for torsion:

$$\tau(s) = (\hat{c}_0 \cdot s + \hat{c}_1)^{-1/\beta}$$

- Four-point interpolation (similar to the 2D algorithm)



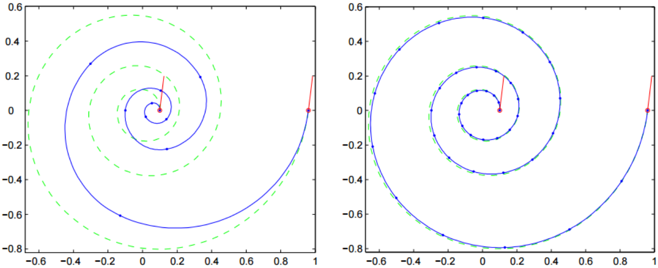


[MF06] K. T. Miura, M. Fujisawa, J. Sone, K. G. Kobayashi, *The Aesthetic Space Curve*. In: Humans and Computers, pp. 101–106, 2006.

[YF09] N. Yoshida, R. Fukuda, T. Saito, *Log-Aesthetic Space Curve Segments*. In: SIAM/ACM Joint Conference on Geometric and Physical Modeling, pp. 35–46, 2009.

Logarithmic arc spline [Ya14]

- Input: position & tangent at the endpoints + winding number
- Generates a series of circles
  - [pro] NURBS-compatible / [con] Only G^1-continuous
- Approximates the logarithmic spiral (i.e., $\alpha = 1$)



Approximating a spiral with (a) 10 or (b) 40 circular arcs.

[Ya14] X. Yang, *Geometric Hermite interpolation by logarithmic arc splines*.
In: CAGD 31(9), pp. 701–711, 2014.

Conclusion & open problems

- Log-aesthetic curves are...
  - fair (monotone curvature)
  - a generalization of classic curves
  - non-standard (NURBS approximations exist)
- Extensions include...
  - 3D LA curves
  - G^2 splines capable of inflections
- Open problems:
  - Alternative 3D generalization?
  - Log-aesthetic surfaces?

Car mockup with LA curves.

Any questions?