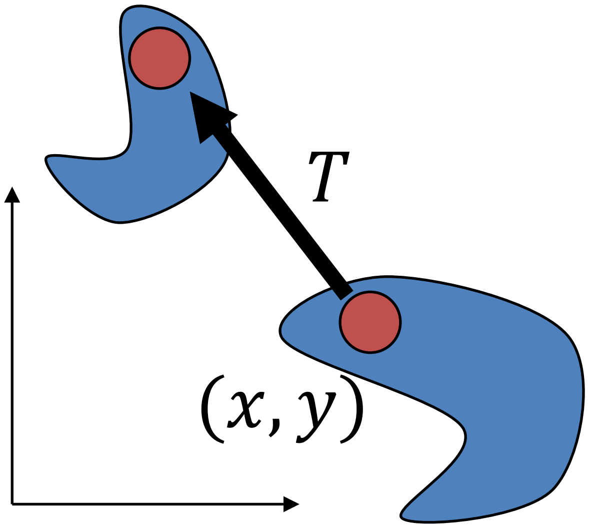
> ”Τὰ πάντα ῥεῖ καὶ οὐδὲν μένει.”
>
> Ἡράκλειτος

Transformations

Szirmay-Kalos László

Transformations

$$(x', y') = T(x, y)$$

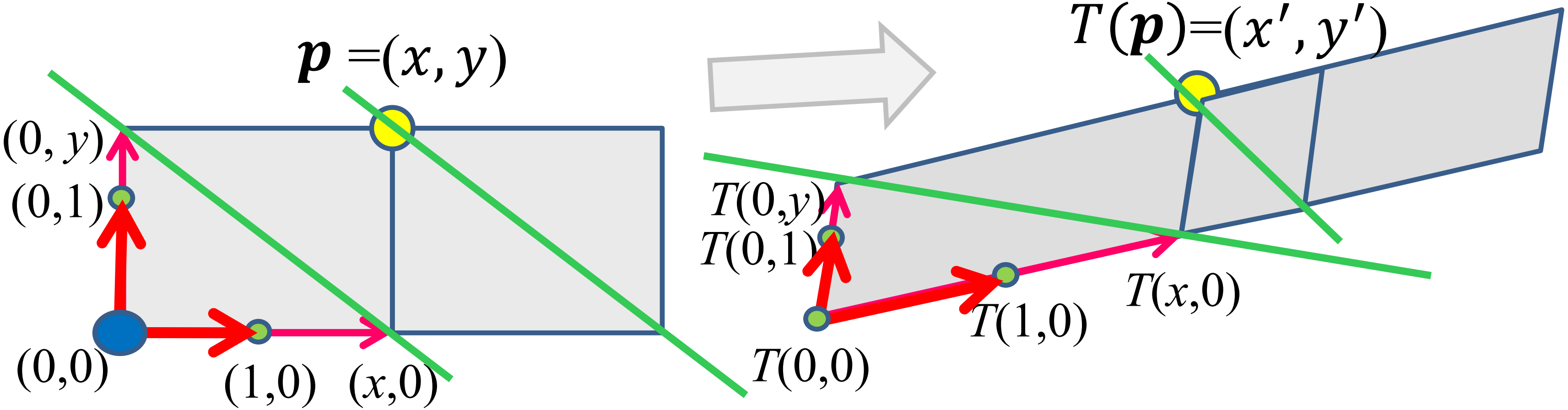


- Assigns points to points
- May destroy the representation and the equation
- Allow transformations preserving lines (segments) and planes (triangles)

Affine transformations

  – Preserves lines and parallelism
  – Translation, rotation, scaling, shearing, reflection…

Affine transformations: preserves parallel lines



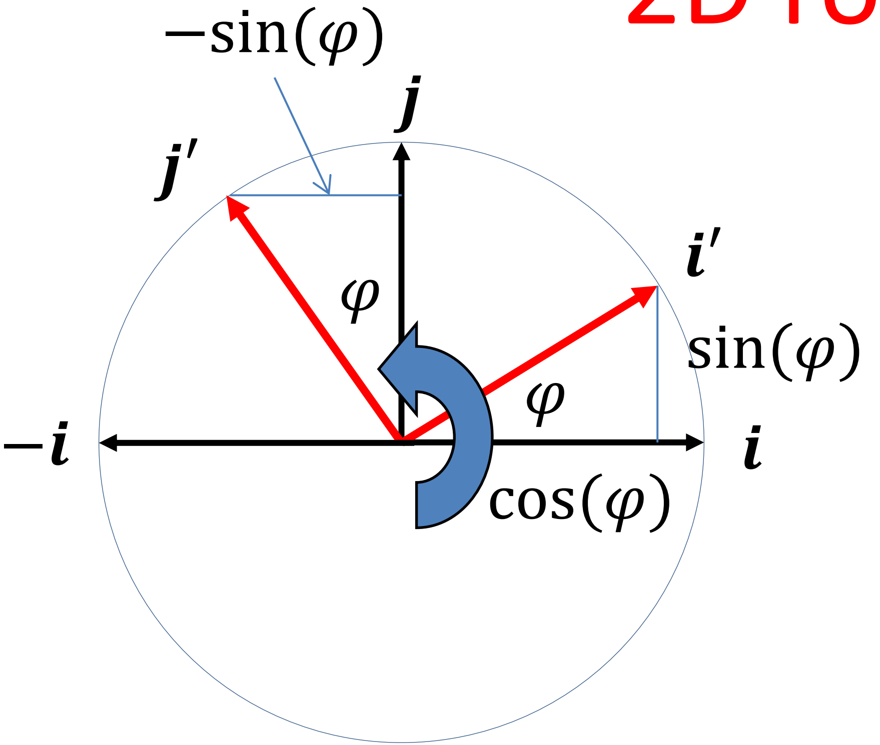
$$T(\boldsymbol{p}) = T(0,0) + \big(T(x,0) - T(0,0)\big) + (T(0,y) - T(0,0))$$

$$= T(0,0) + x\big(T(1,0) - T(0,0)\big) + y(T(0,1) - T(0,0)) \;\; = \boldsymbol{o}' + x\boldsymbol{i}' + y\boldsymbol{j}'$$

$$[x', y', 1] = [x, y, 1] \begin{bmatrix} \boldsymbol{i}'_x & \boldsymbol{i}'_y & 0 \\ \boldsymbol{j}'_x & \boldsymbol{j}'_y & 0 \\ \boldsymbol{o}'_x & \boldsymbol{o}'_y & 1 \end{bmatrix}$$

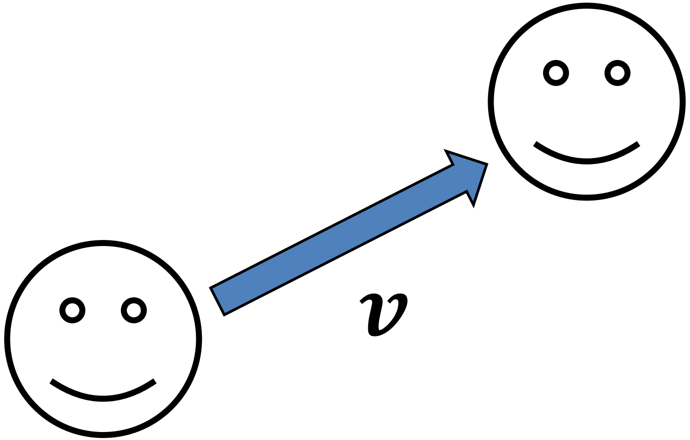
$$x' = ax + by + c$$
$$y' = dx + ey + f$$

2D rotation



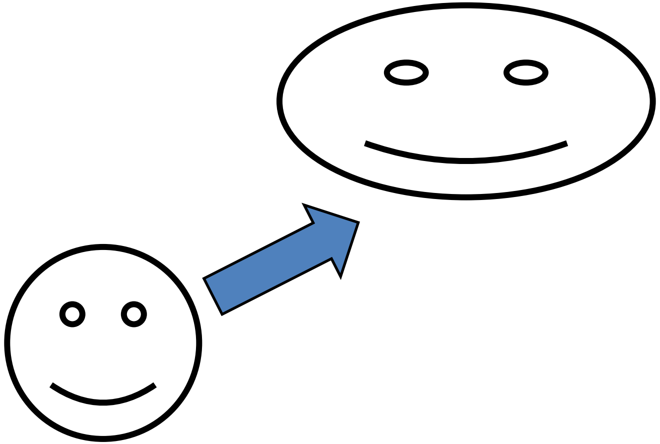
$$[x', y', 1] = [x, y, 1]\begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D translation



$$[x', y', z', 1] = [x, y, z, 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v_x & v_y & v_z & 1 \end{bmatrix}$$

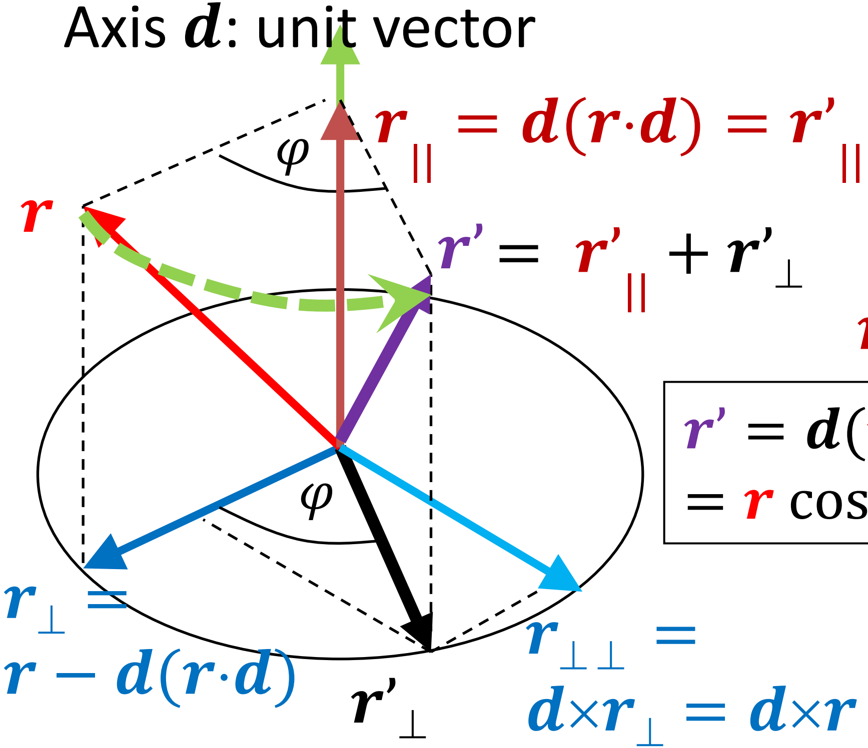
3D scaling



$$[x', y', z', 1] = [x, y, z, 1]\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around axis *d:* Rodrigues formula

Axis $\mathbf{d}$: unit vector



$$\mathbf{r}_{\parallel} = \mathbf{d}(\mathbf{r}\cdot\mathbf{d}) = \mathbf{r}'_{\parallel}$$

$$\mathbf{r}' = \mathbf{r}'_{\parallel} + \mathbf{r}'_{\perp}$$

$$\mathbf{r}_{\perp} = \mathbf{r} - \mathbf{d}(\mathbf{r}\cdot\mathbf{d})$$

$$\mathbf{r}'_{\perp}$$

$$\mathbf{r}_{\perp\perp} = \mathbf{d}\times\mathbf{r}_{\perp} = \mathbf{d}\times\mathbf{r}$$

$$\boxed{\begin{aligned}\mathbf{r}' &= \overset{\mathbf{r}'_{\parallel}}{\mathbf{d}(\mathbf{r}\cdot\mathbf{d})} + \overset{\mathbf{r}_{\perp}}{(\mathbf{r} - \mathbf{d}(\mathbf{r}\cdot\mathbf{d}))}\cos(\varphi) + \overset{\mathbf{r}_{\perp\perp}}{\mathbf{d}\times\mathbf{r}}\sin(\varphi) \\ &= \mathbf{r}\cos(\varphi) + \mathbf{d}(\mathbf{r}\cdot\mathbf{d})(1-\cos(\varphi)) + \mathbf{d}\times\mathbf{r}\sin(\varphi)\end{aligned}}$$

<u>Rows of the matrix:</u>

Images of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and the origin

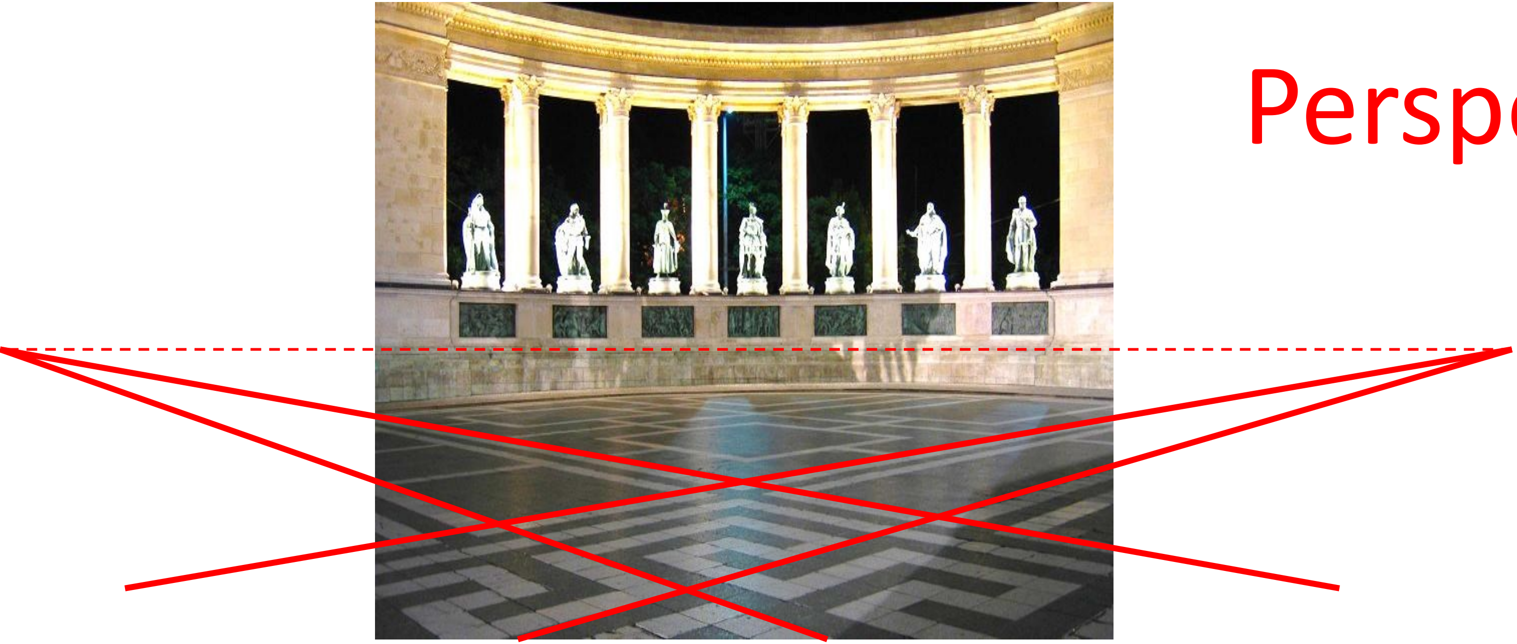
”μὴ εἶναι βασιλικὴν ἀτραπὸν ἐπί γεωμετρίαν”

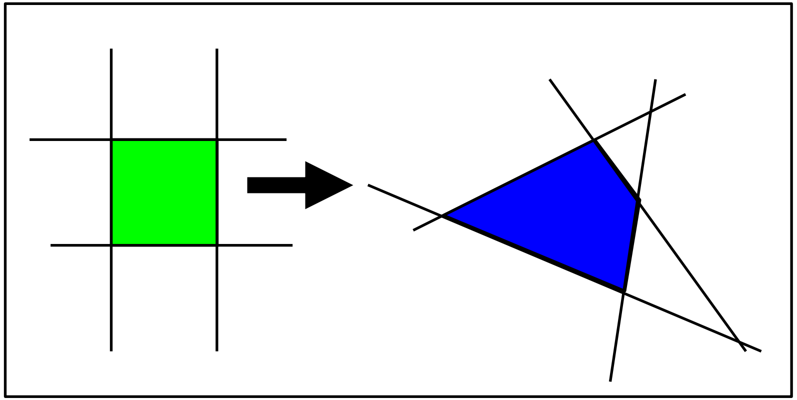
Εὐκλείδης

Projective geometry

Szirmay-Kalos László

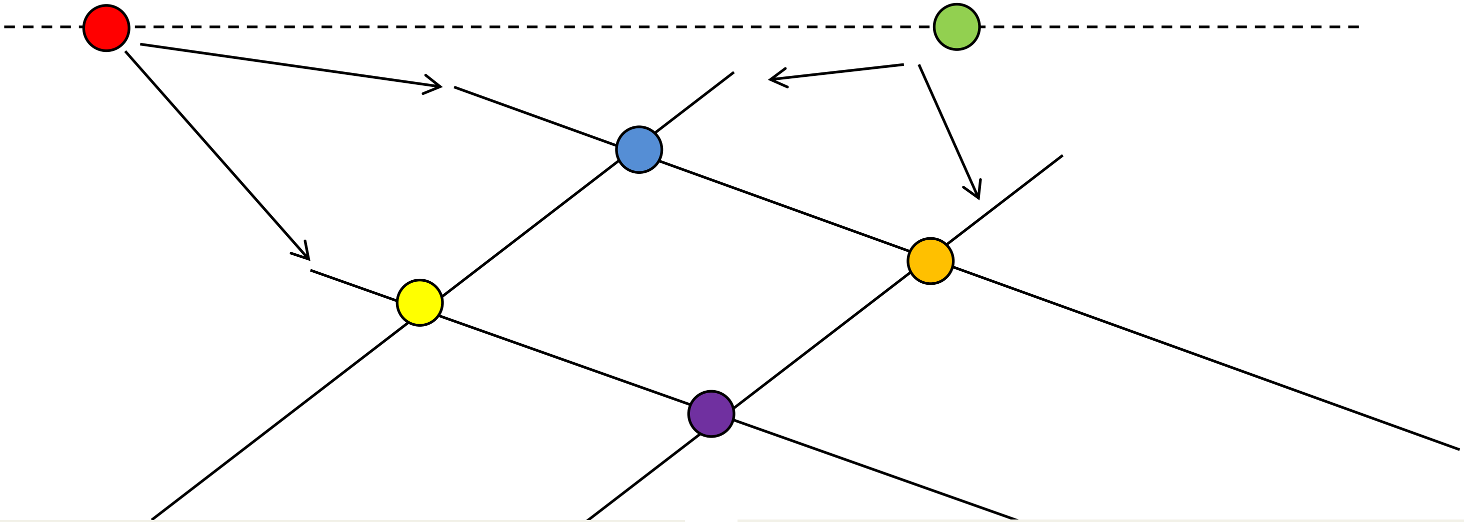
Perspective





- Maps lines to lines
- Does not preserve parallel lines
- Euclidean geometry has a hole

Euclidean → Projective plane



Euclid's axioms:

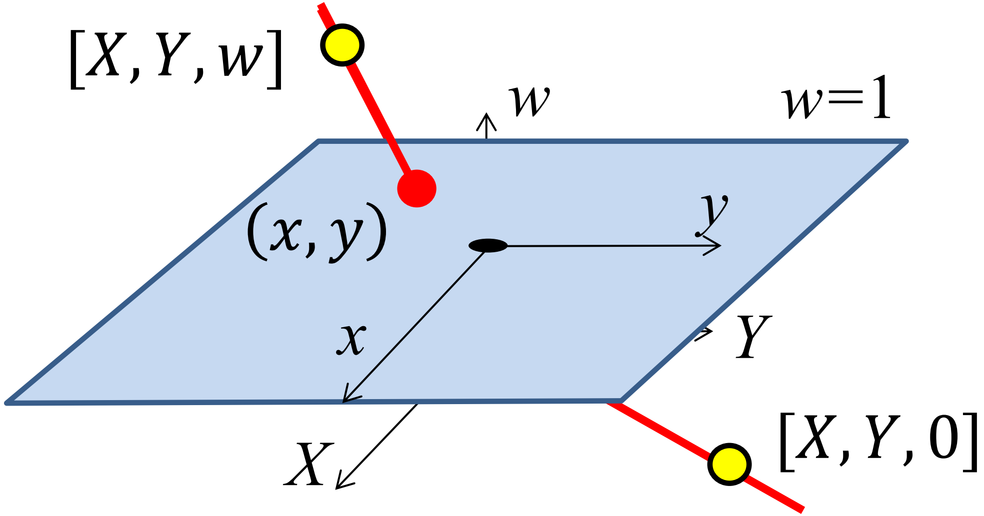
- Two points define a line.
- A line has at least two points.
- ~~If a is a line and A is a point not on this line, then there is exactly one other line that crosses point A but not line a.~~

→

Axioms of projective geometry:

- Two points define a line.
- A line has at least two points.
- **Two lines intersect in a single point**.

Analytic geometry of the projective plane



Euclidean points:

$$(x, y) \rightarrow [x, y, 1] \sim [x \cdot w, y \cdot w, w] = [X, Y, w]$$

Homogenous division: $x = \frac{X}{w}, \; y = \frac{Y}{w}$ $\boxed{w \neq 0}$

Ideal points:

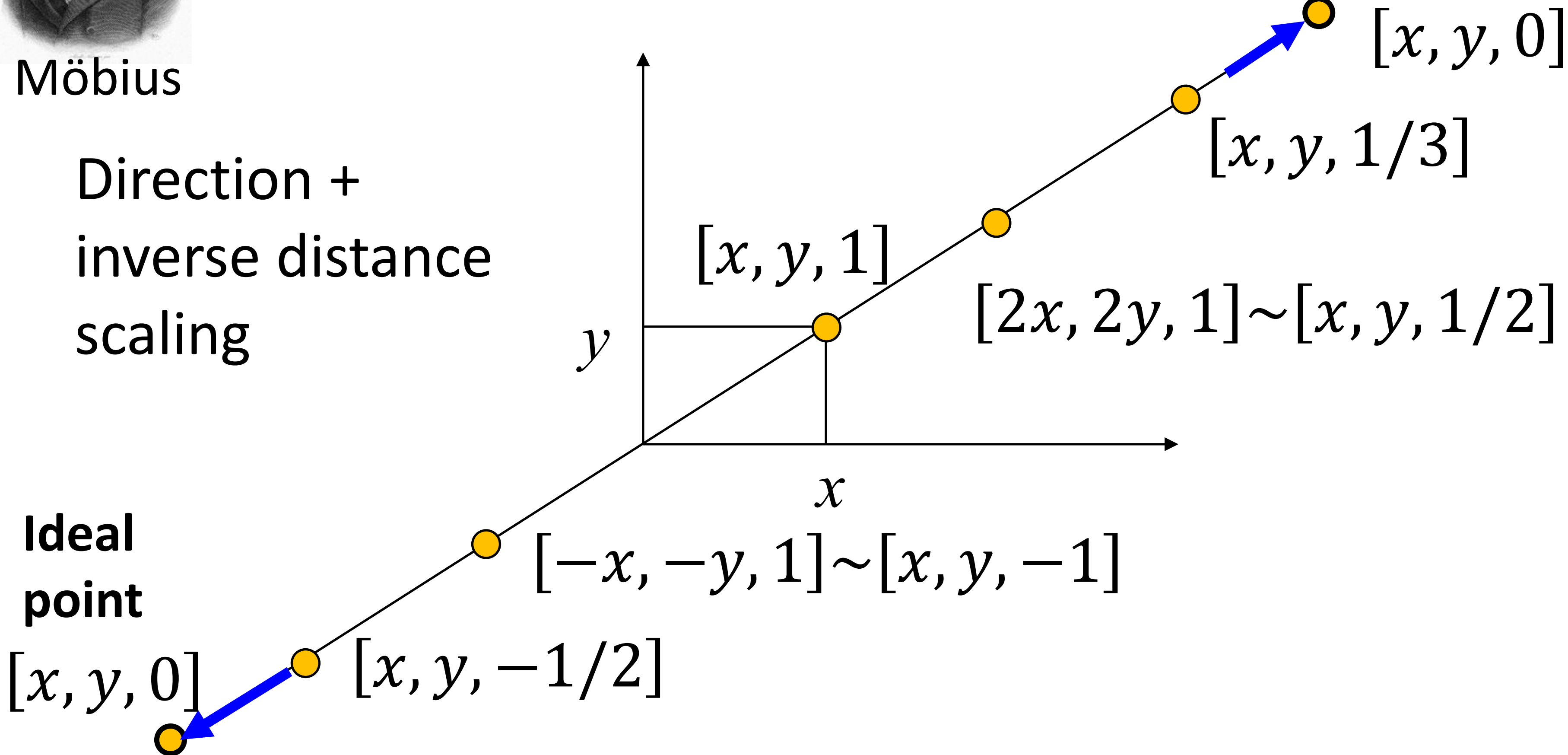
$$[X, Y, 0]$$

Homogenous coordinates

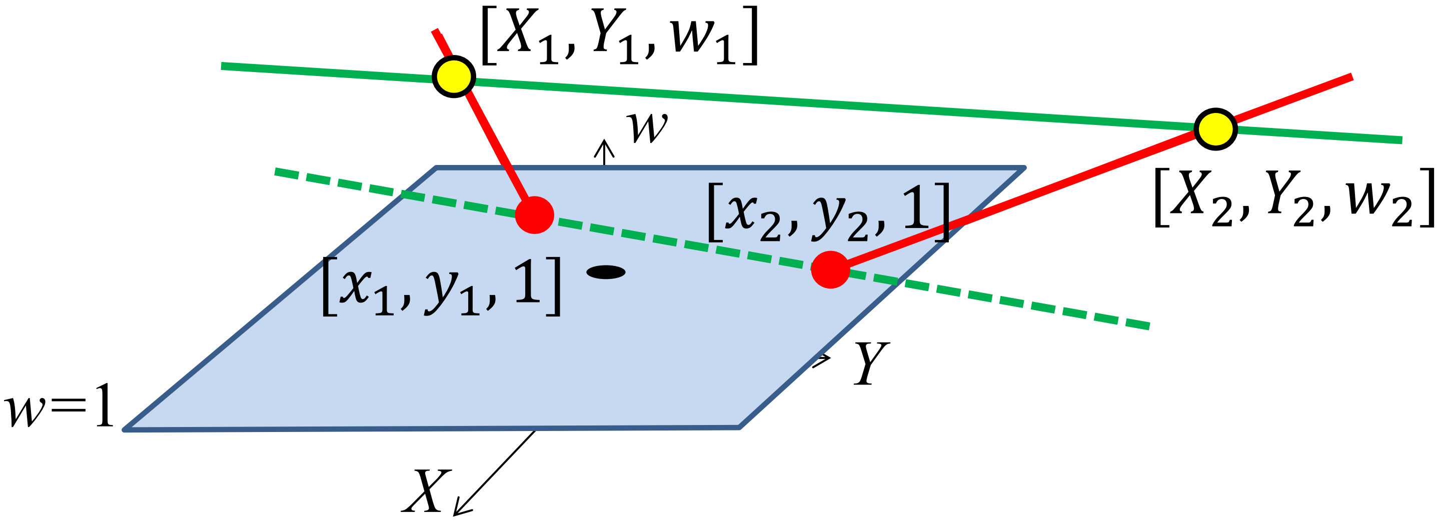
Möbius

Direction + inverse distance scaling

Ideal point



Parametric equation of the Projective Line



$$[X(t), Y(t), w(t)] = [X_1, Y_1, w_1](1-t) + [X_2, Y_2, w_2]t$$

Implicit equation of the line

Euclidean line, Descartes coordinates:

$$n_x x + n_y y + c = 0$$

Euclidean line, homogeneous coordinates:

$$n_x X/w + n_y Y/w + c = 0 \qquad w \neq 0$$

Projective line:

$$\boxed{n_x X + n_y Y + cw = 0} \qquad w \xcancel{\neq} 0$$

Point: row vector

$$[X, Y, w]\begin{bmatrix} n_x \\ n_y \\ c \end{bmatrix} = 0$$

Line: Column vector

Projective space with homogeneous coordinates

- Euclidean points:

  $(x, y, z) \rightarrow [x, y, z, 1] \sim [x \cdot w, y \cdot w, z \cdot w, w] = [X, Y, Z, w]$

  Homogeneous division: $x = \frac{X}{w},\ \ y = \frac{Y}{w},\ \ z = \frac{Z}{w}$

- Ideal points: $[X, Y, Z, 0]$
- Parametric equation of the line:

  $[X(t), Y(t), Z(t), w(t)] = [X_1, Y_1, Z_1, w_1](1 - t) + [X_2, Y_2, Z_2, w_2]t$

- Implicit equation of the plane:

  $n_x X + n_y Y + n_z Z + dw = 0$

Homogeneous linear transformations

Multiplication of homogeneous coordinates by a matrix

- Contains affine transformations
- 2D transformation is a 3×3 matrix

$$[X', Y', w'] = [X, Y, w] \cdot \boldsymbol{T}_{3\times3}$$

- 3D transformation is a 4×4 matrix

$$[X', Y', Z', w'] = [X, Y, Z, w] \cdot \boldsymbol{T}_{4\times4}$$

- Concatenation of transformations: Associative

$$\begin{aligned}[X', Y', Z', w'] &= (\ldots([X, Y, Z, w] \cdot \boldsymbol{T}_1) \cdot \boldsymbol{T}_2) \ldots \cdot \boldsymbol{T}_n \\ &= [X, Y, Z, w] \cdot (\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 \cdot \ \ldots \cdot \boldsymbol{T}_n) \\ &= [X, Y, Z, w] \cdot \boldsymbol{T}\end{aligned}$$

Properties of homogeneous linear transformations

- If invertible: Lines to lines, combinations to combinations, convex combinations to convex combinations
- If not invertible, degeneration possible

Example: lines to lines:

$$[X(t), Y(t), Z(t), w(t)] = [X_1, Y_1, Z_1, w_1]t + [X_2, Y_2, Z_2, w_2](1-t)$$

$$\boldsymbol{p}(t) = \boldsymbol{p}_1 t + \boldsymbol{p}_2(1-t) \;\; // \cdot \mathbf{T}$$

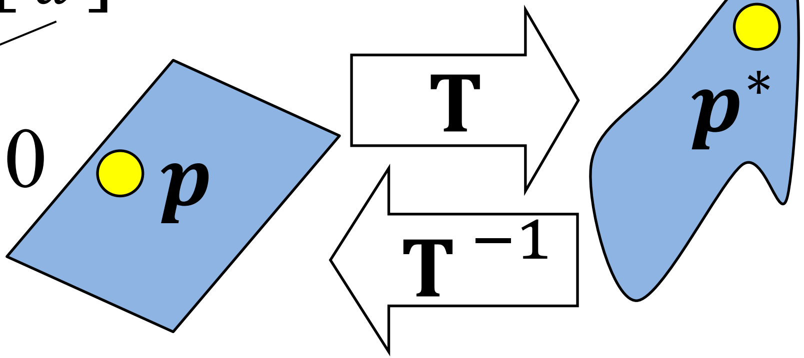
$$\boldsymbol{p}^*(t) = (\boldsymbol{p}_1 \cdot \mathbf{T})t + (\boldsymbol{p}_2 \cdot \mathbf{T})(1-t)$$

$$\boldsymbol{p}^*(t) = \boldsymbol{p}_1^* t + \boldsymbol{p}_2^*(1-t)$$

Invertible transformations: planes to planes

$$[X, Y, Z, w]\begin{bmatrix} n_x \\ n_y \\ n_z \\ d \end{bmatrix} = 0$$

$$\boldsymbol{p} \cdot \boldsymbol{n}^T = 0$$



$$\boldsymbol{p}^* = \boldsymbol{p} \cdot \mathbf{T}$$

$$\boldsymbol{p}^* \cdot \mathbf{T}^{-1} = \boldsymbol{p}$$

$$(\boldsymbol{p}^* \cdot \mathbf{T}^{-1}) \cdot \boldsymbol{n}^T = 0$$

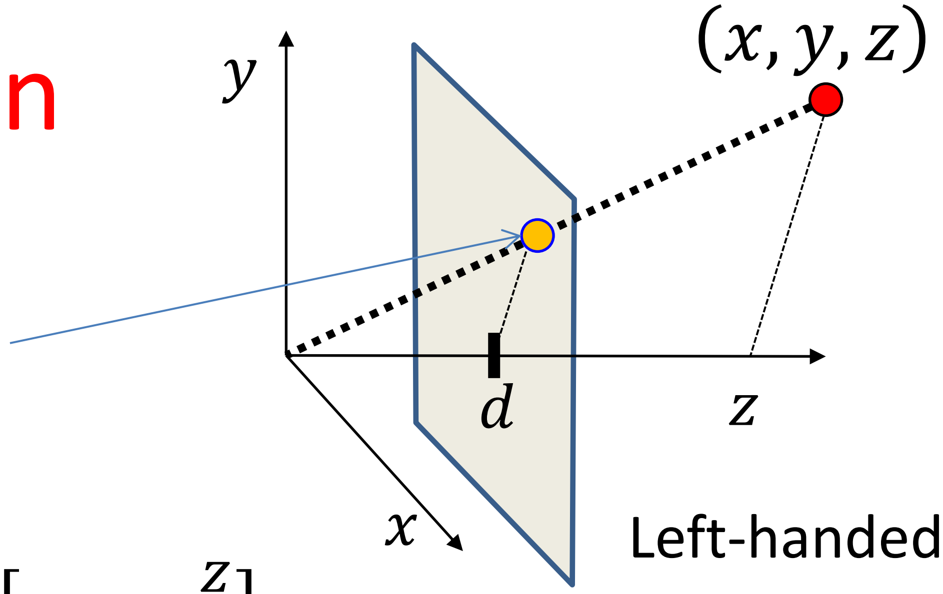
$$\boldsymbol{p}^* \cdot (\mathbf{T}^{-1} \cdot \boldsymbol{n}^T) = 0$$

$$\boldsymbol{p}^* \cdot \boldsymbol{n}^{*T} = 0$$

Transformed plane:

$$\boxed{\boldsymbol{n}^{*T} = \mathbf{T}^{-1} \cdot \boldsymbol{n}^T}$$

Central projection



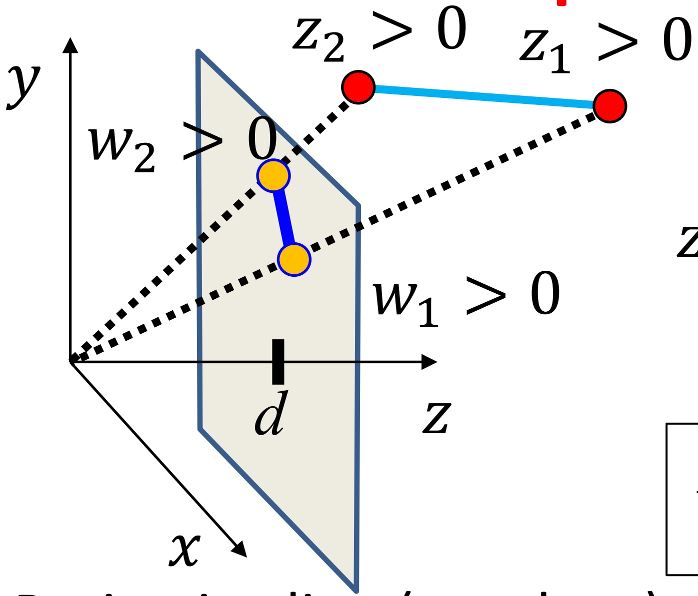
Left-handed

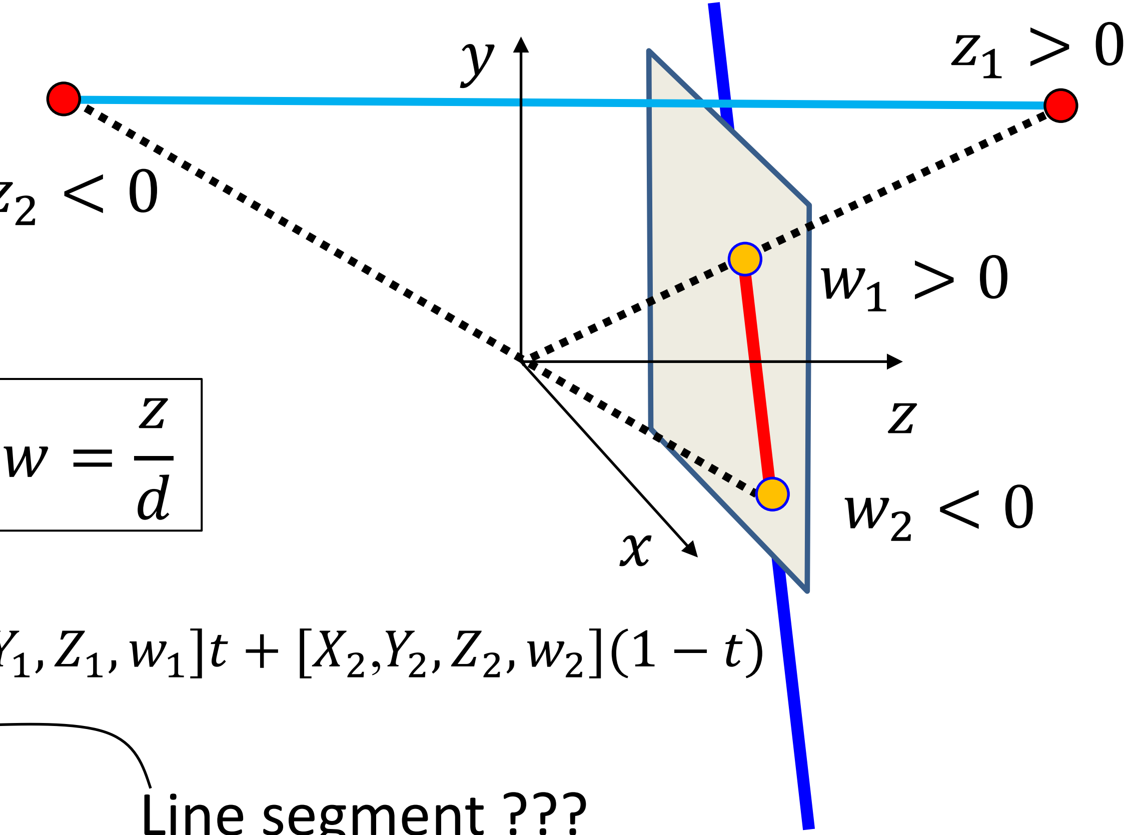
$$(x', y', z') = \left(x\frac{d}{z}, y\frac{d}{z}, d\right)$$

$$[x', y', z', 1] = \left[x\frac{d}{z}, y\frac{d}{z}, d, 1\right] \sim \left[x, y, z, \frac{z}{d}\right]$$

$$[x, y, z, 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/d \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[x, y, z, \frac{z}{d}\right] \sim [x', y', z', 1]$$

Wrap-around problem



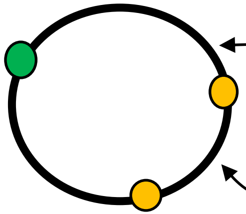


$$\boxed{w = \frac{z}{d}}$$

Projective line (topology)

$$[X(t), Y(t), Z(t), w(t)] = [X_1, Y_1, Z_1, w_1]t + [X_2, Y_2, Z_2, w_2](1 - t)$$

Ideal point



Line segment ???